

The Lagrange points - more than simply solutions to a differential equation

Tasks and solutions

Exercise 1

Research: What are satellites and space probes used for, and which flight paths have advantages for each?

Depending on the scope of functions we can distinguish between many different types of satellite:

Satellite	Exercise
News satellites	Transmitting news
Earth observation satellites	Observing the weather (weather satellites), the Earth in a military sense (spy satellites), and also for scientific purposes.
Television satellites	Transmitting television programmes directly to the viewer.
Astrometry satellites	Observing the Universe for scientific purposes.
Killer satellites	Destroying other satellites.
Research satellites	Carrying out scientific research assignments.
Spy satellites	Spying on e.g. enemy states, ship movements and monitoring arms control agreements. They are operated by military authorities and secret services and are often top secret projects.
Space stations	For the most part they are also satellites which primarily serve scientific purposes.

The flight path of an earth satellite (*orbit*) is selected according to its mission profile: Observation satellites should fly as low as possible so that they can also recognise many details. Spy satellites sometimes even fly so low that friction with the atmosphere limits the lifespan to a few months. On the other hand, communication satellites are supposed to remain stationary above the Earth; for this reason they orbit the Earth at a considerable distance on a *geo-stationary path* (see exercise 2).

Exercise 2

Many people have heard of the so-called *geo-stationary orbit* of a satellite: A satellite flies above the Earth's equator on a path that is synchronous with the Earth's rotation, meaning that it moves at the same rotational speed as the Earth. Calculate at what height such a satellite must fly above the Earth's surface.

A geo-stationary satellite flies around the Earth on a circular path with the radius r . There is an equilibrium of forces between the radial force and the gravitational force:

$$F = m a = m \omega^2 r = G \frac{Mm}{r^2}$$

$$r = \sqrt[3]{G \frac{M}{\omega^2}}$$

Using the angular velocity ω of the Earth or of the geo-stationary satellite

$$\omega = 2\pi f = 2\pi \frac{1}{T}$$

yields

$$r = \sqrt[3]{\frac{G}{4\pi^2} M T^2}$$

Using R as the radius of the Earth, the height h above the Earth's surface equals:

$$h = r - R = \sqrt[3]{\frac{G}{4\pi^2} M T^2} - R = 35\,786 \text{ km}$$

$$G = 6,674 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

$$M = 5,9736 \cdot 10^{24} \text{ kg}$$

$$T = 23 \text{ h } 56 \text{ m } 4,09 \text{ s} = 86\,164,09 \text{ s}$$

$$R = 6371 \text{ km}$$

Exercise 3

Determine the true speed of a point on the Earth's surface in connection with the geographical latitude φ of its location. Compare these values with those of other planets in our Solar System (e.g. the gas giants). What do you notice?

The true velocity v of a point on the Earth is:

$$v = \frac{s}{t} = \frac{2\pi R \cos \varphi}{T} = 0,46 \text{ km/s} \cdot \cos \varphi$$

The planets in our Solar System yield the following true velocities at their equators: ($\varphi = 0^\circ$):

Mercury (0.003 km/s), Venus (0.002 km/s), Earth (0.46 km/s) Mars (0.24 km/s), Jupiter (12.7 km/s), Saturn (10.3 km/s), Uranus (2.6 km/s), Neptune (2.7 km/s).

You will notice that the gas giants rotate far more quickly than the Earth-like planets.

Exercise 4

Carry out Newton's lunar calculation yourself.

Due to the Earth's gravitational force, the apple accelerates uniformly when falling to the Earth which has the radius R_{Earth} . The Moon too is perpetually falling towards the Earth for precisely the same reason: it remains in its orbit with the radius r instead of flying off at a tangent. For this reason, radial and gravitational acceleration must behave inversely to the squares of the distances, exactly as for the forces of attraction.

$$a_{Moon} = \omega^2 r = \frac{4\pi^2}{T^2} r = 2,72 \cdot 10^{-3} \text{ m/s}^2$$

Moon:

Orbital period of the Moon $T = 27.32 \text{ d}$

Distance between Moon and Earth $r = 3.83 \cdot 10^8 \text{ m} = 60 R_{Earth}$

Apple:
$$a_{Appel} = 9,81 \text{ m/s}^2$$

Solve for:

$$\frac{a_{Appel}}{a_{Moon}} = \frac{r^2}{R_{Earth}^2}$$

$$\frac{9,81 \text{ m/s}^2}{2,72 \cdot 10^{-3} \text{ m/s}^2} = \frac{(60 R_{Earth})^2}{R_{Earth}^2}$$

$$3600 = 3600$$

This shows that the accelerations behave like the inverse squares of the distances.

Exercise 5:

Derive the law of gravitation assuming that a planet with mass m is moving on a circular path around a central body with mass M . (Tip: 3. Kepler's third law).

The central force of the mass M exerts the following radial force on the body orbiting at angular speed ω with mass m :

$$F_1 = m \omega^2 r = m \left(\frac{2\pi}{T} \right)^2 r$$

According to Kepler's third law $\frac{T^2}{r^3} = C$:

$$F_1 = m \frac{4\pi}{C r^3} \cdot r = C_1 \frac{m}{r^2} .$$

C_1 is independent of m . According to the interaction principle the orbiting body also exerts an equal and opposite force on the central body by the amount F_2 , which must be correspondingly proportional to mass M of the body on which the force is exerted:

$$F_2 = C_2 \frac{M}{r^2}$$

C_2 is independent of M . If action equals reaction: $\vec{F}_1 = -\vec{F}_2$, i.e. the amounts are the same:

$$F_1 = F_2 = C_1 \frac{m}{r^2} = C_2 \frac{M}{r^2}.$$

Because C_2 is independent of M , C_1 must be directly proportional to M !

$$C_1 = G M.$$

G is called the gravitational constant. Its value is $G = 6.672 \cdot 10^{-11} \text{ m/kg s}^2$. This results in:

$$F_1 = F_2 = C_1 \frac{m}{r^2} = G M \frac{m}{r^2} = G \frac{Mm}{r^2}.$$

Exercise 6:

The comet probe Rosetta is supposed to reach a terminal velocity of $v_E = 3.545 \text{ km/s}$ in outer space. At what speed does Rosetta need to be launched from Earth?

If we assume an escape velocity of $v_2 = 11.2 \text{ km/s}$, the Ariane 5 must attain a speed of

$$v = \sqrt{v_2^2 + v_E^2} = \sqrt{11,2^2 + 3,545^2} = 11,734 \text{ km/s}$$

Therefore an extra speed of approximately 0.534 km/s is sufficient for the probe to attain a final speed of 3.545 km/s .

Exercise 7:

Think about when it is that Lissajous figures yield closed paths! Try to make non-closed figures using the pendulum experiment!

Lissajous figures are curve graphs that are created by superimposing two linear harmonic oscillations that are perpendicular to one another. How it looks depends on the frequency ratio and the phase angle difference from the starting point. If the two frequencies are in a rational ratio to each other the Lissajous figure does not change and the path is closed. If not, with time the Lissajous loop will cover the entire surface.

Exercise 8:

Research on the internet which satellites and spacecraft have already been sent to the libration points of the Sun-Earth system. What scientific reasons for this can be found in the mission profiles?

Information can be found on the internet under the following links:

- [ICE \(ISEE 3\)](#)
- [GENESIS](#)
- [SOHO](#)
- [JAMES WEB SPACE TELESCOPE \(JWST\)](#)
- [DARWIN/TPF](#)
- [WMAP](#)
- [Herschel](#)
- [Planck](#)